Error Analysis

Introduction

An understanding of the analysis of errors which arise in experiments is an essential part of understanding the meaning of the results of such experiments. Errors in scientific experiments generally are classed in one of three categories: mistakes, systematic errors and random errors.

Mistakes sometimes occur, regardless of how careful or honest the experimenter. They have one good characteristic--they are correctable. Once they are discovered, one can redo the experiment, or redo the analysis of the data so that a correct result is obtained.

More insidious are the systematic errors. They result from faults in equipment or unknown effects upon the equipment. It is frequently extremely difficult to detect the presence of such problems. However, almost any equipment may contain such errors. As simple a device as a meter stick, stamped out of incompletely dried wood, may give a systematically incorrect reading. It is then no surprise that highly complex instruments, such as particle accelerators, may have built into them systematic problems that it may take a very long time to find. In some situations, experimenters arbitrarily quote errors as much as 10 times greater than those obtained directly from the data, in order to allow for such systematic errors.

The real errors, errors which are absolutely unavoidable, are the random errors. These are errors that enter the data in a random way, from unknown effects, but which cause results which are off from the true result as much in one direction as in the other. These are the errors which one can handle by analyzing the actual measurements, and on the analysis of which one therefore spends a large amount of time.

Accuracy and Precision

Before starting to discuss the details of handling errors, a few other things should be mentioned. The two terms accuracy and precision are sometimes used interchangeably, but they have different specific meanings in the context of error analysis. The term accuracy refers to how closely the result of a measurement reproduces the "true" value of the quantity measured. The precision, on the other hand, refers to the self-consistency and reproducibility of the data itself. It is probably easier to see the distinction by using an example. The accepted value of the speed of light is about 3.0 X 108 m/sec. Suppose that a measurement of c were made and the result (2.53±.01) X 108 m/sec was obtained. One would conclude that this measurement was very precise but not very accurate. The spread in the data is relatively small, only about ±1 part in 250, but the result differs from the accepted value by almost 20%. In practice, one strives for precision in experiments, and hopes that systematic errors do not seriously affect the accuracy of the measurement.

Mean, Standard Deviation, and Standard Error (see Appendix for details)

It will be the case that in any data that you take, there will be some errors that creep in due to mistakes, systematic errors in instrumentation, and unknown reasons. You should quantify the errors in your measurements by providing three important pieces of information which quantify your measurements and the errors in them. To obtain these quantities, you should try to take as much data as possible, which means repeating your experiment as many times as feasible within the time you have.

The first piece of information that is important is the mean (also called average, expectation value, etc.) and is something you are probably familiar with. If you make 100 measurement of the length of a string, and you get 100 values, then you simply add up all the values and divide by the number of data points, in this case 100. The mean tells you what is the length of the string on average.

Say the measurements are . The mean of these measurements, is given by

, *i = 1,2,3……100*

Next, you should determine the standard deviation. Let us stick with the 100 measurements for length of a string; to obtain the standard deviation, subtract the mean from each of the 100 measurements. Some numbers will now be positive and others negative since your mean may be greater or less than a particular measurement of the length. If you square the deviations, you will have a positive number. Add up the squares of the deviation and divide by the number of data points, 100 in this case, and you get the variance. The square root of the variance is the standard deviation. Note that this definition of standard deviation is accurate only when you take a large number of measurements. If you take a large number of measurements, say 100 measurements, the standard deviation will be about the same as if you took a 1000 measurements. So, the standard deviation gives you information about the quality of the data. The formula for standard deviation,  is,

, i = 1, 2, 3……100

Once you have the standard deviation, divide it by the square of the number of measurements you made, 100 in this case, and you get the standard error. Note that the standard error is quite different from the standard deviation. The former decreases as you collect more data, but the latter does not change since it is an intrinsic property of the data you are collecting. The formula for the standard error, *S.D*., if you have 100 data points, is

*S.D.* =

Significant figures

We strongly recommend that you report your data and results with appropriate significant figures, and that when you have to round off numbers, use rules that are consistent. The appendix gives you more information on significant figures and rounding. In this course, you will not be penalized for non-usage or incorrect usage of significant figures and rounding, but it is to your benefit to know what the generally accepted practices are so that when needed in your later courses or career, you will know what to do and how to do it.

**APPENDIX**

Mean and Standard Deviation

In what follows, we will assume N measurements were taken in the attempt to determine the length (x) of the object of interest. A graph of N vs x might look like:



We assume that, in the process of measuring some quantity which we will label x, there have been obtained N values of x, labelled xi. Not all of the xi are necessarily distinct. In fact, in most cases, if N is fairly large, some values of xi will occur many times in the sample, while others will occur rarely or not at all. Then one of the most important parameters that can be derived from this sample is the mean, which is defined as



Once this quantity has been calculated, one may calculate the deviations for each data point:

i = xi -

Two important properties of the deviations follow. If the sample is large enough the deviations will be positive about as often as they are negative. Then:

Δ = = ≅ 0

A non-zero quantity is obtained by averaging the magnitudes of the deviations:



The last expression is clearly not zero. However, it is also not of much use in analyzing things. For analytical purposes, it is much easier, and amounts to much the same thing, to use the squares of the deviations. The procedure is then to square all the deviations, positive or negative, and then add the squares:

2 =

= - 2 + N 2

We can now define a quantity called the standard deviation, for which the usual symbol is , and it is defined by the equation

2 = 2

= 2

= 

Notice that this equation expresses the square of the standard deviation, which is referred to as the *variance* of the distribution.[[1]](#footnote-1)

There is a problem with the expression for the standard deviation as given above. The problem is that the definition assumes that is the true mean of an infinite set of measurements. However, in any real experiment is actually based on a finite sample of measurements. For example, suppose that we make a single measurement of x. If we then try to estimate  by using , we get 2  0, and since N=1, we get = 0, which is unrealistic, at best. The number of data points (N) represents what is called the number of degrees of freedom of the sample of data points, which reflects the fact that each of the data points was obtained independently of the others. If we now impose some sort of restriction on the data, we reduce the number of degrees of freedom by 1. Therefore, the number of degrees of freedom is N‑1, and that means that in the equation for the standard deviation, we should be dividing by N-1 instead of N. Standard notation replaces  by s when N-1 is used:

s2 = 2

Note that

s2 = 2

so that as N->, s2 -> 2 . Thus, as might be expected, the distinction between s and  disappears as more and more data points are taken.

Standard Error: The standard error, S.E., is a common and useful way of quantifying the error in measurements and is defined as the ratio of the standard deviation, , to the number of measurements, N. Thus,

S.E. = /sqrt(N)

What this simple expression tell you is that if you want reduce the error by a factor of 2, you have to take 4 times more data!

In your labs, you should try to take as much data as you can, since the standard error varies inversely with the amount of data. Make sure you report the mean, associated standard deviation, and standard error for all quantities you measure.

Significant figures and rounding

One of the more confusing aspects of recording data arises when deciding how many digits to record, particularly after some mathematical manipulation has taken place. In writing the value of a measurement, one must decide how well the number is really known. For instance, if you measure the length of an object with a meter stick, you might be able to measure to an accuracy of a millimeter without much effort. You would then want to record the length in such a way as to reflect this precision. You need to determine how many "significant digits" there are in the number to be recorded. Apart from the placement of the decimal point, the symbols that carry real information are the significant digits. The meaning of significance is seen if you imagine that the length that you measured is 23.5 cm and you are confident that this result is precise. In the performance of a calculation, you might have to multiply this length by, say, . Your calculator might give you the result 73.827, which looks like a very precise result. However, the last two digits are useless, because the original measurement was only accurate to three digits. If the next place were to be measured, and that result multiplied by  also, the last two digits would likely vary considerably. Hence you are really certain only of the first three digits.

The first thing to learn in writing significant digits is when a digit in a number is significant and when it is not. Consider the three numbers

123,000.

123 x 103

1.230 x 105

Each of these has a different number of significant digits.

The rules are as follows:

1) The left-most non-zero digit is always the most significant digit, regardless of how the number is written. Thus whether you write 0.000123 or 1.23 x 10-4, the most significant digit is the 1. All the zeros to the left of the first non-zero digit do nothing to the significance of the number. They merely place the decimal point.

2) If no decimal point is written, the right-most non-zero digit is the least significant. Thus, if you write 123,000 the least significant digit is 3. The zeroes to the right of the 3 simply place the decimal point, which is not written in this case. The significance is the same if you write this number in the form 123 x 103 or 1.23 x 105.

3) If the decimal point is written, then the right-most digit is significant, regardless of where it is. Thus, writing 123,000. tells you that there are 6 significant digits, the right-most 0 being the least significant.

The safest way to be sure that the number of significant digits is clear is to write numbers using an exponential form. Any number can be expressed as a number between 1 and 10 multiplied by a suitable power of 10. Thus the number used just above can be written 1.23 x 105 if there are 3 significant digits, or 1.23000 x 105 if there are 6 significant digits.

Another somewhat delicate point involves how to round a number. Frequently, in the process of calculation with experimental results, extra digits are generated, for instance during a multiplication or division (as was shown in the example above). In quoting a result, it is improper to indicate more significant digits than are justified by your measurements, so it is necessary to round off the excess digits. The rules that are usually followed are

1) If the last digit is less than 5, simply drop it. If the last digit is greater than five, increment the previous digit. Thus 1.2344 is rounded to 1.234, but 1.2346 is rounded to 1.235.

2) If the last digit is exactly 5, drop it if the previous digit is even. If the previous digit is odd, increment it. A brief way of expressing this is: round even. The rationale for this procedure is that roughly half the time, the number will be incremented and half the time it will not. Any error introduced by this procedure should tend to cancel out. As examples, the number 1.2345 would become 1.234, but the number 1.2355 would become 1.236.

3) If it is necessary to round off more than one digit, all the digits to be dropped should be considered at once. The above rules are then followed. Thus, 1.23551 becomes 1.236, and 1.23549 becomes 1.235.

It is possible to lose significance through mathematical calculations. This is of particular importance in subtracting two numbers. If the two numbers are fairly close in value, a very large fraction of the number of significant digits can be lost. For example

3.568

-3.627

-.0.059

In this case the numbers being subtracted both had four significant figures. The difference left only two! In analysis of data, one should try to avoid such situations. This is, of course, not always possible. The only way out of this problem is to try to obtain even more precise measurements.

*Note: You won’t always lose credit for mistakes in significant figures, but in some cases a point could be deducted.”*

1. The terminology at this point varies somewhat among authors. [↑](#footnote-ref-1)